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Directed random walks on directed percolation clusters

Xiao-Hong Wang
Minerva Center and Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, Israel
and Department of Thermal Science and Energy Engineering, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

Ehud Perlsman and Shlomo Havlin
Minerva Center and Department of Physics, Bar-llan University, 52900 Ramat-Gan, Israel
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The characteristics of directed random walks on directed percolation clusters are numerically studied. For two-dimensional clusters grown at the critical probability p_c, it is shown that the distance d of the directed random walkers from their most probable end point is determined by a probability distribution $p(d) \sim d^{-(1+w/\nu)}$, where the values of w and ν are close to the values of the known exponents: $w \simeq 0.50$ and $\nu \simeq 0.63$. This probability distribution is independent of the cluster's length t up to d values comparable to the cluster's width $\sim t^{\nu}$. The results are shown to be consistent with a tree description of the directed percolation clusters.

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The subject of random walks on percolation clusters, known as the "ant in the labyrinth" problem [1], has drawn quite intensive attention in the physical literature [2]. However, in spite of the numerous efforts, no simple connection was found between the static and dynamical exponents of percolation, though the Alexander-Orbach conjecture [3] comes close to this goal. In this article, it is shown that such a simple connection does exist in the special case of directed random walks on directed percolation clusters.

In the directed percolation model [4], there is a predefined direction in which the percolation clusters can grow. In the directed random walk considered in the present study, walks are also performed only in the same predefined direction. Balents and Kardar [5] have already studied random walks on directed percolation clusters, but while their main concern was the variability of distances between *different clusters*, the main concern of the present article is the variability of distances *within each cluster*.

In order to study this issue, numerical results were obtained for directed random walks performed along two-dimensional directed percolation clusters grown at the critical probability p_c. These numerical results indicated that the distance d of the directed random walkers from their most probable end point is determined by a probability distribution $p(d) \sim d^{-(1+w/\nu)}$, where $w \simeq 0.50$ and $\nu \simeq 0.63$. This probability distribution is independent of the cluster's length t up to d values comparable to the cluster's width $\sim t^{\nu}$. The known values of these two exponents are shown to be consistent with a tree description of the directed percolation clusters.

The article starts with a description of the model and the way the numerical data were gathered. The second part presents the numerical results which have similar characteristics to previous results [6] obtained for the directed polymer model [7,8]. This similarity is investigated further in the final part of the article. The connection between the directed percolation model and the directed polymer model was studied in a number of articles [9–13], and the present study points to some similarity between entropy in the directed percolation model, and energy in the directed polymer model.

On an arbitrary lattice structure, a directed percolation cluster can be defined as the set of lattice points (sites) which are connected to some origin, while the connecting paths are oriented in a predefined direction (thus all of them are of *equal length*). The bonds (or sites) which compose the possible paths are assigned at random from a bimodal (0,1) distribution, and a connecting bond is one whose value is zero. The probability to get a zero-valued bond is denoted by p, and there is a critical probability (p_c) to get a zero-valued bond, above which the probability to get infinite directed percolation clusters is positive. In the two-dimensional case at p_c, the width of the clusters and the distance of their center from the origin grow in proportion to t^{ν}: $\nu \simeq 0.633$ [14], where t is the length of the cluster.

In this study, random bond directed percolation clusters were grown at the critical probability p_c using the Leath method [15]. The clusters were grown to various lengths: more than a million clusters were grown for $t = 32\,000$, and more than 10 000 were grown for $t = 256\,000$. For each cluster point (site), the number of paths connecting it to the origin was recorded. At the cluster's length t, the point with the highest number of connecting paths is denoted by (m,t). This point is the most probable end point of a directed random walker which survives t steps without being trapped in one of the dead ends of the cluster. Of course, in each cluster the position of the point (m,t) is different, and the mean transversal distance of the point (m,t) from the origin is $\sim t^{\nu}$.

On the regular lattice, the point $(0,t)$ is the point with the highest number of paths to the origin at (0,0), a fact which makes it the most probable end point of a directed random walker after t steps. Thus, just as the transversal distances in the regular lattice are measured from the point $(0,t)$, the transversal distances in the present case were measured from the point (m,t). Two main measures for the distance d of the random walkers from the point (m,t) were used in this study. The first is the ABS variable, which measures the mean dis-

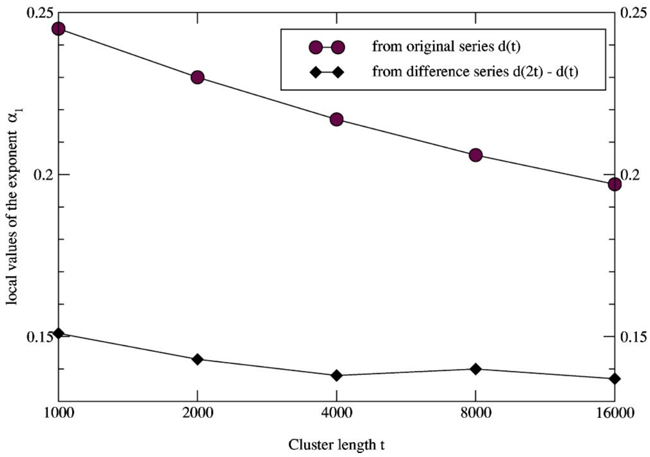

FIG. 1. The local values of the exponent α_1 as a function of the cluster length t. The upper curve is derived from the original data series $d(t)$, and the lower curve is derived from the difference data series $d(2t)-d(t)$.

tance, and the second is the RMS variable, which measures the root of the mean squared distance. In order to overcome the problem of finite-size effects, the difference series, $d(2t)-d(t)$, was also computed. It is easy to see that if $d(t)\sim t^{\alpha}$, then $d(2t)-d(t)$ should also grow as t^{α}, and the results related to the difference series might converge faster to the asymptotic behavior of $d(t)$.

The results of the numerical study are presented in terms of local values of the growth exponents computed by $\log_{\lambda^2}(V(\lambda t)/V(t/\lambda))$, where V is the variable whose rate of change is estimated, and λ is an arbitrary constant: $\lambda=2$ in this study.

As mentioned above, if a random walker is actually sent to a directed walk along a directed percolation cluster, it is most probable that he will stop at one of the dead ends of the cluster. Thus, the present study is limited to random walks which survive t steps, and arrive at the base of a t length directed percolation cluster. The distance of the random walkers from the point (m,t) is expected to grow as t^{α}, and Figs. 1 and 2 present the local values of the exponents α_1 and α_2 related to the ABS and RMS distances. The data are presented both for the exponents computed from the series $d(t)$ and from the difference series $d(2t)-d(t)$. The data presented in these figures lead to the estimates $\alpha_1=0.138(3)$, $\alpha_2=0.386(3)$. Note that while both measures characterize the mean transversal distance from the origin, they scale with t in a significantly different rate.

Both α_1 and α_2 are determined by the probability distribution $p(d,t)$, which expresses the probability that the random walker will be found at a distance d from the point (m,t) after t steps. The significant difference between the two exponents indicates that $p(d,t)$ should have an unusual form, and as shown in the following, after an initial phase, and up to certain values of d, the form of $p(d,t)$ is $\sim d^{-k}$. The local values of the exponent k computed for $t=32\,000$ and $256\,000$ are presented in Fig. 3, from which it is possible to estimate $k=1.76(3)$.

This form of the probability distribution implies that most of the random walks end at a bounded region around the

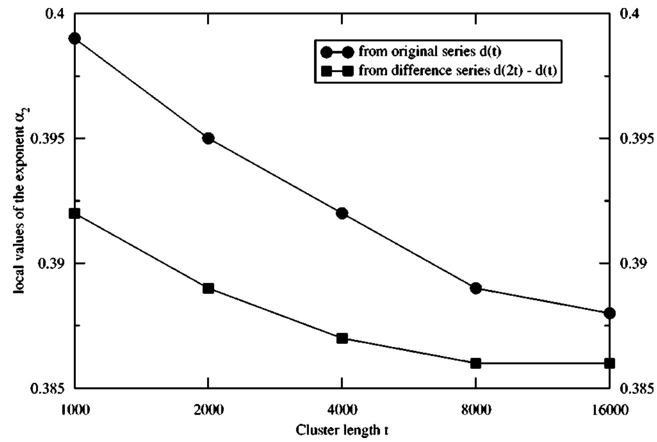

FIG. 2. The local values of the exponent α_2 as a function of the cluster length t. The upper curve is derived from the original data series $d(t)$, and the lower curve is derived from the difference data series $d(2t)-d(t)$.

point m: Even for infinite clusters, 95% of the walks end at a distance $d<128$, and 99% of them end at a distance $d<1024$.

It is easy to see that for a probability distribution of the form

$$p(d,t)\sim d^{-k},\quad d_0<d<t^{\nu}\ \ k>1,$$

$$p(d,t)=0,\quad d>t^{\nu}$$

the moments of the probability distribution, defined as $M_n\equiv\langle d^n\rangle^{1/n}$, have the values $M_n\sim t^{\nu(n+1-k)/n}=t^{\nu-(k-1)\nu/n}$. Let us define the exponent $w\equiv\nu(k-1)$; the exponent related to the nth moment is thus $\alpha_n=\nu-w/n$. Assuming this form of probability distribution holds in the present case, the estimates obtained from Figs. 1 and 2 ($\alpha_1\simeq0.138$, $\alpha_2\simeq0.386$) lead to the estimates $w\simeq0.496$, $\nu\simeq0.634$. The

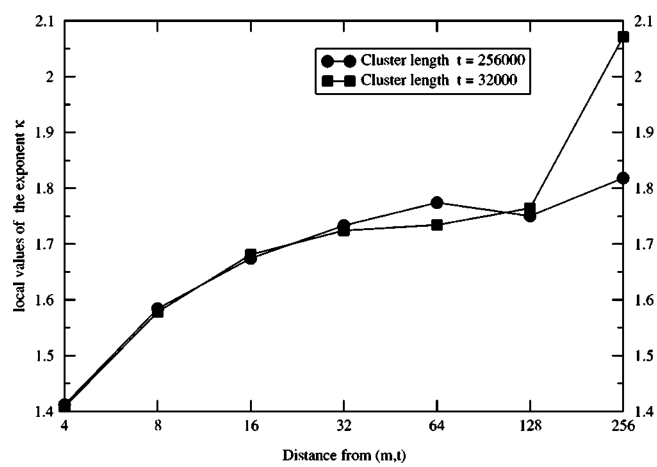

FIG. 3. The local values of the exponent k as a function of the distance from the point (m,t). The circles denote the results derived from clusters of length $t=256\,000$, and the squares denote the results derived from clusters of length $t=32\,000$. Note that in both curves the values obtained for $d=256$ deviate from the asymptotic behavior due to the small size of the clusters.

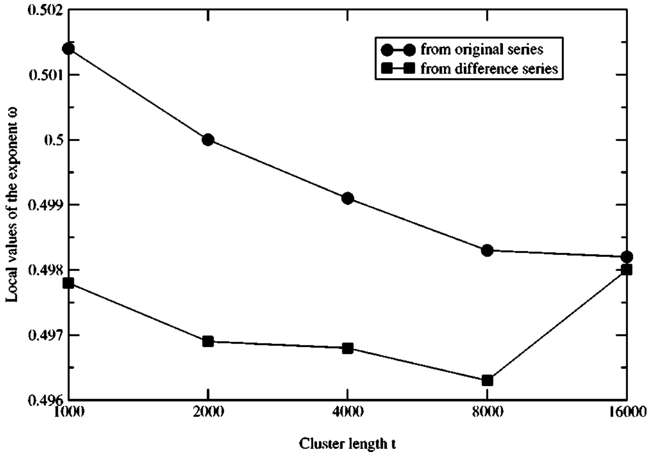

FIG. 4. The local values of the exponent w as a function of the cluster length t. The upper curve is derived from the original data series $\log N(t)$, and the lower curve is derived from the difference data series $\log N(2t) - \log N(t)$.

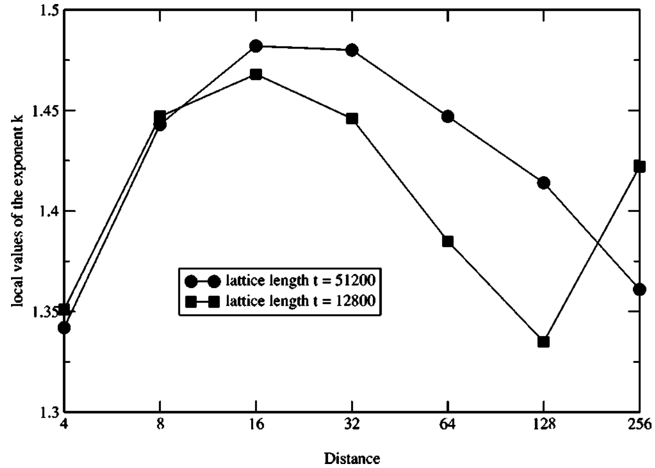

FIG. 5. The local values of the exponent k as a function of the distance between the end point of the optimal path and the end point of the second-best path. The circles denote results obtained for lattices whose $t = 51\,200$, and the squares denote results obtained for lattices whose $t = 12\,800$.

third, fourth, and fifth moments of the distribution were also computed from the numerical data, and the resultant estimates are $\alpha_3 = 0.467(3)$, $\alpha_4 = 0.509(3)$, $\alpha_5 = 0.535(3)$. All these estimates are consistent with the above estimates of ν and w. The estimate of $k = 1.76(3)$ is also consistent with the relation $k = 1 + w/\nu$.

Of course, the estimate $\nu \simeq 0.634$ is very close to the direct estimate of $\nu \simeq 0.633$ [14], while the exponent $w \simeq 0.496$ is shown in the following to be the exponent related to the variability between the logarithms of the number of connecting paths (configurations) of the clusters. Figure 4 presents the local values of the exponent w computed from the original series (upper curve) and the difference series (lower curve). The resultant estimate is $w = 0.497(1)$, in agreement with the former estimate of Balents and Kardar [5]: $w = 0.50(1)$. All the results obtained for the moments of the distribution are also consistent with the values $\nu = 0.633$ and $w = 0.497$.

Passing from the numerical results to their explanation, it should be noted that the purpose of the explanation is to link the exponents w and ν with the exponent k through the relation $k = 1 + w/\nu$. The explanation presented in the following relies heavily on the similarity between the numerical results obtained in the present case and those obtained in [6] for the directed polymer model. The variable which was referred to in [6] is the distance between the end points of the best path (least energy path) and the second-best path of the regular lattice. It was shown in [6] that the ABS distance between these points grows as $t^{\nu - w}$, where ν is the space exponent and w is the energy exponent of this model: $\nu = \frac{2}{3}$ and $w = \frac{1}{3}$ in the two-dimensional case.

From the discussion presented in [6], it is easy to conclude that the nth moment of the distribution $\sim t^{\nu - w/n}$, which implies that in the two-dimensional case, $p(d) \sim d^{-k}$: $k = 1.5$. In order to verify this conclusion, more than 100 000 random lattices were used to compute the local values of the exponent k for $t = 12\,800$ and $51\,200$. The results are presented in Fig. 5, and though finite-size effects are significant both for $d < 16$ (because d is small) and for $d > 32$ (because t is not large enough), it is clear that the maximal local value of the exponent k approaches the value 1.5, and that as t increases, this maximal value extends for larger d values, in agreement with the presumed value of $k = 1 + w/\nu = 1.5$ derived above.

The similarity between the results obtained for the directed polymer model and those presented in Figs. 1–3 suggests that the factors which determine both sets of results might also be similar. Following the theoretical derivation of the results presented in [6], with some necessary adaptations, the numerical results presented in Figs. 1–3 can be explained using the following arguments.

(i) It is possible to divide the whole cluster to a set of subclusters. A subcluster which splits at height h from the one leading to (m,t) is found at a distance $d \sim h^{\nu}$ from (m,t), and its width is also $\sim h^{\nu}$. (Actually, in the directed polymer model, the optimal paths of the regular lattice form a tree structure, while in the directed percolation cluster there are also loops. The presence of loops makes it impossible to strictly follow the more exact derivation presented in [6].)

(ii) Denote by N_c the total number of configurations of the cluster, by $N_{sc}(d)$ the number of configurations of the subcluster which surrounds the point at distance d from (m,t), and by $n_c(d)$ the number of configurations leading to that point (referred to as "the point d" in the following). The probability $p(d)$ is thus computed as $\langle n_c(d)/N_c \rangle$. Note that in many random clusters $n_c(d) = 0$, since the probability of the point d to belong to the directed percolation cluster is < 1 and decreases with d.

(iii) Two numerical findings are utilized in the following. The first is that the probability distribution of $\log N_c(t)$ is a regular (close to Gaussian) distribution whose $\sigma \sim t^{w}$ (see Fig. 4). The second is that $N_{sc}(0) \simeq N_c$ (see Fig. 3 and the discussion related to that figure). The relation $N_{sc}(0) \simeq N_c$ can also be derived in an analytic way, and this issue is discussed further below.

(iv) Multiply and divide $n_c(d)/N_c$ by $N_{sc}(d)$ to get

$$p(d)=\langle[n_c(d)/N_{sc}(d)]*[N_{sc}(d)/N_c]\rangle$$
$$=\langle n_c(d)/N_{sc}(d)\rangle*\langle N_{sc}(d)/N_c\rangle.$$

The last equality follows from the fact that d might be any point of its subcluster, and thus there is no correlation between the two (multiplied) ratios.

(v) Since the point d might be any point of its subcluster, the mean ratio $\langle n_c(d)/N_{sc}(d)\rangle$ is inversely proportional to the width of its subcluster. This width is shown in point (i) above to be proportional to d, and thus $\langle n_c(d)/N_{sc}(d)\rangle \sim 1/d$.

(vi) Turn now to the mean ratio $\langle N_{sc}(d)/N_c\rangle$, and note that $N_{sc}(0)\simeq N_c$ [see point (iii) above]. Denote by p^* the probability that $N_{sc}(d)>N_{sc}(0)$. Since in these cases $N_{sc}(d)/N_c\simeq 1$, the following relation holds:

$$\langle N_{sc}(d)/N_c\rangle \simeq p^*+(1-p^*)*\langle N_{sc}(d)/N_{sc}(0)\rangle,$$

where the last mean is computed for the cases in which $N_{sc}(0)>N_{sc}(d)$.

(vii) Relating to the cases in which $N_{sc}(0)>N_{sc}(d)$, the mean ratio $\langle N_{sc}(d)/N_{sc}(0)\rangle$ can be computed by $\int_{-\infty}^{0}p(y)e^y$, where $y\equiv\log(N_{sc}(d)/N_{sc}(0))$. The two subclusters (of 0 and d) can be treated as two independent clusters whose length is $\sim d^{1/\nu}$ [see point (i) above], and thus the probability distribution of y is a Gaussian whose $\sigma_y\simeq d^{w/\nu}$ [see point (iii) above]. It is easy to verify that for $\sigma_y \gg 1$, the value of this integral is $\simeq 1/\sigma_y \simeq d^{-w/\nu}$.

(viii) In order to calculate p^*, note that p^* is the area below the probability distribution of $y \equiv\log(N_{sc}(d)/N_{sc}(0))$, computed for positive values of this variable. Since the probability distribution of this variable is continuous, in both sides of zero its height is $\simeq 1/\sigma_y \simeq d^{-w/\nu}$. Remember that $N_{sc}(0)\simeq N_c$ [see point (iii) above], and thus the maximal value of $\log(N_{sc}(d)/N_{sc}(0))$ is bounded. From this it follows that p^* is also proportional to the height of the Gaussian near zero $\simeq d^{-w/\nu}$. This result, together with the one derived in point (vii) above, implies that $\langle N_{sc}(d)/N_c)\rangle \sim d^{-w/\nu}$.

(ix) The above arguments lead to the conclusion that the probability of the random walker to arrive at a distance d from (m,t) is $\sim 1/d*d^{-w/\nu}\sim d^{-(1+w/\nu)}$. Naturally, this relation can hold only up to d values comparable to the cluster's width $\sim t^{\nu}$.

One might wonder why $\langle n_c(d)/N_{sc}(d)\rangle$ is inversely proportional to the width of the subcluster, while $\langle n_c(d)/N_c\rangle$ is not inversely proportional to the width of the whole cluster $\sim t^{\nu}$. The answer is that the point d might be the point with the highest number of configurations in its subcluster, while it cannot be the point with the highest number of configurations in the whole cluster, which is by definition the point whose $d=0$. The above derivation is possible because the point (m,t), whose position is different in each random cluster, is chosen as the point from which distances are measured.

The above derivation relies on the numerical result $N_{sc}(0)\simeq N_c$. Actually, this result can also be proved using the same line of arguments presented above: Note that if $p(d)\sim d^{-k};k>1$, then $N_{sc}(0)\simeq N_c$. Note also that apart from the replacement of "$\simeq$" by "$<$" in some of the relations above, the only argument which relies on the result $N_{sc}(0)\simeq N_c$ is the one related to p^* in point (viii) above. However, since the point with the highest number of configurations in the subcluster of d has fewer configurations than the point whose $d=0$, it is clear that $\log(N_{sc}(d)/N_{sc}(0))<\log d$, and thus $p^*<d^{-w/\nu}\log d$, which completes the proof.

Let us define by $d_{1,2}$ the distance between the point (m,t) and the point with the second highest number of connecting paths to the origin. The discussion presented above leads to the conclusion that the distance $d_{1,2}$ has similar characteristics to those of the random walker distance from (m,t). This conclusion was verified in the numerical study, and the estimates of the relevant α_1 and α_2 are 0.138(5) and 0.383(1), respectively. These estimates are very close to those obtained for the random walks' distance.

In conclusion, the numerical results obtained in this study indicate that the distance of the random walkers from the point (m,t) is characterized by the probability distribution $d^{-(1+w/\nu)}$, where $w\simeq 0.497$ and $\nu\simeq 0.633$. The moments of this distribution $\langle d^n\rangle^{1/n}$ grow in proportion to t^{α_n}, where $\alpha_n=\nu-w/n$, and all the numerical results are consistent with these values. The results are explained assuming a tree structure of the directed percolation clusters, and though in these clusters there are also loops, their existence had no observable influence on the numerical results obtained in the study.

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